Relation between roots and coefficient of Polynomial equation.

Kirit Vaniya
Department of Mathematics
M. G. Science Insitute

For polynomial of power 2

- Let $ax^2 + bx + c$ is a polynomial equation of degree 2
- Let α & β are roots of given polynomial.
- Hence α & β satisfies the polynomial equation

$$ax^2 + bx + c = 0 \qquad \dots\dots\dots eq.(1)$$

Since the equation whose roots are α & β is given by

$$(x-\alpha)(x-\beta) = 0$$

$$\Rightarrow x^2 - (\alpha+\beta)x + (\alpha\beta) = 0 \qquad \dots\dots.eq.(2)$$

- From equation 1 we have,

$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{c}{a}\right) = 0 \qquad \dots\dots\dots eq.(3)$$

Comparing eq.2 and 3 we get following relations.

$$(\alpha+\beta) = -\left(\frac{b}{a}\right) \ \& \ \alpha\beta = \left(\frac{c}{a}\right) \qquad \dots eq.(I)$$

For polynomial of power 3

- Let $ax^3 + bx^2 + cx + d$ is a polynomial equation of degree 3
- Let α, β & γ are roots of given polynomial.
- Hence α & β satisfies the polynomial equation

$$ax^3 + bx^2 + cx + d = 0 \qquad \ldots\ldots\ldots eq.(1)$$

Since the equation whose roots are α, β & γ is given by

$$(x-\alpha)(x-\beta)(x-\gamma) = 0$$

$$\Rightarrow x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - (\alpha\beta\gamma) = 0 \qquad .eq.(2)$$

- From equation 1 we have,

$$x^3 + \left(\frac{b}{a}\right)x^2 + \left(\frac{c}{a}\right)x + \left(\frac{d}{a}\right) = 0 \qquad \ldots\ldots\ldots eq.(3)$$

Comparing eq.2 and 3 we get following relations.

$$(\alpha+\beta+\gamma) = -\left(\frac{b}{a}\right), \alpha\beta+\beta\gamma+\gamma\alpha = \left(\frac{c}{a}\right), \alpha\beta\gamma = -\left(\frac{d}{a}\right) \ldots eq.(I)$$

For polynomial of power 4

- Do the same procedure and derive the relations.

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$if\ \alpha, \beta, \gamma\ \&\ \delta\ are\ roots\ then\ we\ have\ following\ relations.$$

$$(\alpha + \beta + \gamma + \delta) = -\left(\frac{b}{a}\right)$$

$$\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \beta\delta + \alpha\gamma = \left(\frac{c}{a}\right)$$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha = -\left(\frac{d}{a}\right)$$

$$\alpha\beta\gamma\delta = \left(\frac{e}{a}\right)$$

Example-1: if sum of two roots of the equation

$$x^3 - 4x^2 - 9x + 36 = 0$$

is zero then solve the equation.

Solution-1:

Let α, β & γ are roots of given polynomial equation,

Also let $\alpha + \beta = 0$ ….(1)

We know that the relation between the roots and coefficient of polynomial of degree 3 are given as follows.

$$(\alpha + \beta + \gamma) = -\left(\frac{b}{a}\right) = -4 \ \dots (2)$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \left(\frac{c}{a}\right) = -9 \dots (3)$$

$$\alpha\beta\gamma = -\left(\frac{d}{a}\right) = 36 \dots (4)$$

Solution-1:

We have following equations.

$$\alpha + \beta = 0 \quad \dots (1)$$

$$(\alpha + \beta + \gamma) = -4 \quad \dots (2)$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -9 \dots (3)$$

$$\alpha\beta\gamma = 36 \quad \dots (4)$$

Using equations 1 and 2 we get,

$$0 + \gamma = \gamma = -4 \quad \dots (5)$$

Also from equation 1 we get $\alpha = -\beta \quad \dots (6)$

Using equation 5, 6 in 4 we get following

$$(-\beta)(\beta)(-4) = 36$$

$$\Rightarrow \beta^2 = 9$$

$$\Rightarrow \beta = \pm 3 \quad \dots (7)$$

Hence using equation 6 we get

$$\alpha = \mp 3 \quad \dots (8)$$

Hence the required three roots are $3, -3 \; and \; 4$

$$(\alpha, \beta, \gamma) = (-3,3,4) \; or \; (3,-3,4)$$

Example-2: if ratio of two roots of the equation

$$4x^3 - 16x^2 + 9x + 9 = 0$$

is 1/2 then solve the equation.

Solution-2:

required three roots are $3/2\ , 3\ and\ \ -1/2$

$$(\alpha, \beta, \gamma) = \left(\frac{3}{2}, 3\ , -\frac{1}{2}\right)$$

Example-3: if α, β & γ are roots of equation

$$2x^3 + 3x^2 - 4x + 1 = 0$$

then find the equation whose roots are,

$\frac{1}{1-\alpha}, \frac{1}{1-\beta}, \frac{1}{1-\gamma}$

Solution-3:

Taking $y = \frac{1}{1-x}$ and substituting in above equation value of x we get required result.

Solution is $2y^3 - 8y^2 + 9y - 2 = 0$

Example-4: if α, β & γ are roots of equation

$$x^3 - 12x + 16 = 0$$

then find the equation whose roots are

(i) $\alpha(\beta+\gamma), \beta(\alpha+\gamma), \gamma(\alpha+\beta)$

(ii) $(\alpha-\beta)^2, (\beta-\gamma)^2, (\gamma-\alpha)^2$

Solution:

(i) hint : convert $\alpha(\beta+\gamma)$ in only one variable α and call it y and take $\alpha = x$ substitute this relation of x and y in equation.

$y^3 + 24y^2 + 144y + 256 = 0$ is required solution.

(ii) hint : same as in last but use of main equation in deriving the relation in single power.

$$\left(\frac{48}{y-12}\right)^3 - 12\left(\frac{48}{y-12}\right) + 16 = 0$$

Example-5: if roots of the equation

$$6x^3 - 11x^2 - 3x + 2 = 0$$

are in harmonic progression then solve the equation.

Solution:

Hints:

First $\frac{1}{\alpha-d}, \frac{1}{\alpha}, \frac{1}{\alpha+d}$ are roots of given equation.

Do reciprocals and find the equation whose roots are

$$(\alpha - d), (\alpha), (\alpha + d)$$

$$\frac{1}{3}, 2 \; \& -\frac{1}{2}$$

Then solve using relations find $\alpha \; and \; d$ then substitute in

$$\frac{1}{\alpha - d}, \frac{1}{\alpha}, \frac{1}{\alpha + d}$$

Which are required roots. $3 \; , 1/2 \; , -2$

THANK YOU