CARDON'S METHOD TO SOLVE A CUBIC EQUATION

Kirit vaniya

Department of Mathematics

M. G. Science Institute

Introduction

- Consider a cubic equation.

$$ax^3 + 3bx^2 + 3cx + d = 0 \quad \ldots\ldots(1)$$

Now choose $y = ax + b$ $i.e. x = \frac{y-b}{a}$

Replace x by $\frac{y-b}{a}$ in equation (1)

$$a\left(\frac{y-b}{a}\right)^3 + 3b\left(\frac{y-b}{a}\right)^2 + 3\text{c}\left(\frac{y-b}{a}\right) + d = 0$$

$$y^3 + 3(ac - b^2)y + (2b^2 - 3abc + a^2d) = 0$$

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$$y^3 + 3(ac - b^2)y + (2b^3 - 3abc + a^2d) = 0$$

Taking H $= ac - b^2$ and

$$G = 2b^3 - 3abc + a^2d$$

We have following equation

$$y^3 + 3Hy + G = 0 \dots \dots (2) \; where,$$

H $= ac - b^2$ and $G = 2b^2 - 3abc + a^2d$

Now we know that for any $x', y', \& z'$

$$x'^3 + y'^3 + z'^3 - 3x'y'z'$$
$$= (x' + y' + z')(x' + wy' + w^2z')(x' + w^2y' + wz') \; \dots\dots (3)$$

Where w is cube root of unity

Introduction.

- Taking $x' = y, y' = p$ & $z' = q$
- We have $y^3 - 3pqy + p^3 + q^3 = 0$ .....(4)

$$y^3 - 3pqy + p^3 + q^3$$
$$= (y + p + q)(y + wp + w^2q)(y + w^2p + wq) = 0$$

Hence

$$\left.\begin{aligned} (y + p + q) = 0 &\quad \Rightarrow y = -(p + q) \\ (y + wp + w^2q) = 0 &\quad \Rightarrow y = -(wp + w^2q) \\ (y + w^2p + wq) = 0 &\quad \Rightarrow y = -(w^2p + wq) \end{aligned}\right\} (5)$$

Introduction

- Now form equation (2) &(4)

$$H = -pq \;\&\; G = p^3 + q^3$$

Also $H^3 = -p^3q^3\,, G = p^3 + q^3$

Now p^3 & q^3 are roots of equation

$$(t - p^3)(t - q^3) = 0$$

$$t^2 - t(p^3 + q^3) + q^3p^3 = 0$$

$$t^2 - Gt - H^3 = 0 \qquad \ldots.(6)$$

equation (6) is a quadratic equation in variable t we have root of such equation as

$$t = \frac{G \pm \sqrt{\Delta}}{2} \; where \Delta = G^2 + 4H^3$$

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If $\Delta > 0, then\ p\ and$ q are real and distinct

If $\Delta = 0, then\ p\ and\ q$ are real and equal also the corresponding roots of equation (4) are $-2p, p, p$

If $\Delta < 0\ , then\ p\ \&\ q\ are\ complex$.

From equation (5) if one root is $\phi\ i.e. p = \phi\ then\ q = \frac{-H}{\phi}$

Hence roots of given equation are

$$\left.\begin{aligned} y &= -\phi + \frac{H}{\phi}. \\ y &= -w\phi + \frac{w^2 H}{\phi}. \\ y &= -w^2\phi + \frac{wH}{\phi}. \end{aligned}\right\} \quad \ldots(7)$$

Where $w = \frac{-1+\sqrt{3}i}{2}$

& $w^2 = \frac{-1-\sqrt{3}i}{2}$

Corresponding $x = \frac{y-b}{a}$ are required roots

Example-1 solve the equation

$$2x^3 + x^2 + x - 1 = 0$$

using Cardon's method.

- Solution :

Here given equation is $2x^3 + x^2 + x - 1 = 0$ compare the given equation with

$$ax^3 + 3bx^2 + 3cx + d = 0$$

We have $a = 2, b = \frac{1}{3}, c = \frac{1}{3}$ $and\ d = -1$

Taking $x = \frac{y-b}{a}$ we get equation $y^3 + 3Hy + G = 0$

Where $H = ac - b^2 = \frac{5}{9}$

And $G = 2b^3 - 3abc + a^2d = -\frac{124}{27}$

Example-1 solution

- $\Delta = G^2 + 4H^3$

$$= \left(-\frac{124}{27}\right)^2 + 4\left(\frac{5}{9}\right)^3$$

$$= \frac{196}{9} > 0$$

Hence the roots are real and distinct

$$\phi = \sqrt[3]{\frac{G + \sqrt{G^2 + 4H^3}}{2}}$$

$$\phi = \frac{1}{3}$$

Example-1 solution

$\phi = \frac{1}{3}$ taking $w = \frac{-1+\sqrt{3}i}{2}$ & $w^2 = \frac{-1-\sqrt{3}i}{2}$

roots are as follows

$$y = -\phi + \frac{H}{\phi} = \frac{4}{3} \quad \Rightarrow \quad x = \frac{y-b}{a} = \frac{1}{2}.$$

$$y = -w\phi + \frac{w^2 H}{\phi} = -\frac{2}{3} - \sqrt{3}i \quad \Rightarrow \quad x = \frac{y-b}{a} = w^2 .$$

$$y = -w^2\phi + \frac{wH}{\phi} = -\frac{2}{3} + \sqrt{3}i \Rightarrow \quad x = \frac{y-b}{a} = w .$$

Hence required roots are $\frac{1}{2}, w, \& \ w^2$.

Example-2 :$x^3 - 3x^2 + 12x + 16 = 0$

Example-3 : $x^3 + x^2 - 16x + 20 = 0$

Example-4 : $2x^3 - 7x^2 + 8x - 3 = 0$

Example-5 : $x^3 + 6x^2 - 12x + 32 = 0$

Example-2 : solve the equation

$$x^3 - 3x^2 + 12x + 16 = 0$$

using Cardon's method.

- Solution :
- $H = 3$ and $G = 26$
- $\Delta = 784.$
- $\phi = 3$
- $y = -2 \ \ so\ x = -1$
- $y = \left(1 - 2\sqrt{3}i\right)\ so\ x = 2(1 - \sqrt{3}i\ ),$
- $y = \left(1 + 2\sqrt{3}i\right)\ so\ x = 2(1 + \sqrt{3}i\ )$

Example-3 : solve the equation

$$x^3 + x^2 - 16x + 20 = 0$$

using Cardon's method.

- Solution :
- $H = -\frac{49}{4}$ and $G = \frac{686}{27}$
- $\Delta = 0$, so roots are real and equal.
- $\phi = \frac{7}{3}$
- $y = -\frac{14}{3}$ $so\ x = -5$
- $y = \frac{7}{3}$ $so\ x = 2,$
- $y = \frac{7}{3}$ $so\ x = 2.$
- Required roots are $-5, 2, 2.$

Example-4 : solve the equation

$$2x^3 - 7x^2 + 8x - 3 = 0$$

using Cardon's method.

- Solution :
- $H = -\frac{1}{9}$ and $G = \frac{-2}{27}$
- $\Delta = 0$, so roots are real and equal.
- $\phi = \frac{-1}{3}$
- $y = \frac{2}{3}$ so $x = \frac{3}{2}$
- $y = \frac{-1}{3}$ so $x = 1$,
- $y = \frac{-1}{3}$ so $x = 1$.

Example-5 : solve the equation

$$x^3 + 6x^2 - 12x + 32 = 0$$

using Cardon's method.

- Solution :
- $H = -8$ and $G = 72$
- $\Delta = 3136,$
- $\phi = 4$
- $y = -6 \;\; so \; x = -8$
- $y = 3 - \sqrt{3}i \; so \; x = 1 - \sqrt{3}i,$
- $y = 3 + \sqrt{3}i \; so \; x = 1 + \sqrt{3}i$

THANK YOU