FERRARI'S METHOD

Kirit vaniya

For polynomial of power 4

- Suppose we want to solve the equation

$$x^4 + 9 = 0$$

We use the fact that $M^2 - N^2 = (M + N)(M - N)$.

First we note that $(x^2 + 3)^2 = x^4 + 6x^2 + 9$

Hence $x^4 + 9 = x^4 + 6x^2 + 9 - 6x^2$

$$= (x^2 + 3)^2 - \left(\sqrt{6}x\right)^2$$

$$= \left\{x^2 + 3 + \sqrt{6}x\right\}\left\{x^2 + 3 - \sqrt{6}x\right\}$$

Hence,

$$x^4 + 9 = 0 \Rightarrow \left\{x^2 + 3 + \sqrt{6}x\right\}\left\{x^2 + 3 - \sqrt{6}x\right\} = 0$$

$$\left\{x^2 + 3 + \sqrt{6}x\right\} = 0 \; or \; \left\{x^2 + 3 - \sqrt{6}x\right\} = 0$$

Now it is quadratic equation so we can easily solve it.

Ferrari's method for quartic equation

The basic idea behind this method is as discussed in the first example

We will reduce the main quartic(bi-quadratic) equation in two quadratic equation and as method for solution of quadratic equation is known we can easily solve main equation.

Method is as follows.

Let the quartic equation is given as

$$f(x) = ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0 \quad \ldots\ldots\ldots (1)$$

We use the fact that $M^2 - N^2 = 0 \Rightarrow$

$$(M + N)(M - N) = 0 \Rightarrow$$

$$(M + N) = 0 \; or \; (M - N) = 0$$

Ferrari's method for quartic equation

We start with

$$(ax^2 + 2bx + s)^2 - (2mx + n)^2 = 0 \;\; \ldots\ldots\ldots eq(*)$$

For some s, m & n

$$\Rightarrow (a^2x^4 + 4b^2x^2 + s^2 + 4abx^3 + 4bsx + 2asx^2)$$
$$- (4m^2x^2 + 4mnx + n^2) = 0$$
$$\Rightarrow a^2x^4 + 4abx^3 + (2as + 4b^2 - 4m^2)x^2 + (4bs - 4mn)x$$
$$+ (s^2 - n^2) = 0 \;\; \ldots\ldots\ldots (2)$$

By equation (1) we have

$$a * f(x) = a^2x^4 + 4abx^3 + 6acx^2 + 4adx + ae = 0 \;\; \ldots\ldots\ldots (3)$$

Comparing equation (2) and (3) we get

$$2as + 4b^2 - 4m^2 = 6\text{ac}$$
$$4bs - 4mn = 4ad$$
$$s^2 - n^2 = ae$$

So we have

$$as + 2b^2 - 2m^2 = 3\text{ac} \;, bs - mn = ad, s^2 - n^2 = ae \;\; \ldots\ldots\ldots (4)$$

Ferrari’s method for quartic equation

Now we have

$$as + 2b^2 - 2m^2 = 3\text{ac} \ldots\ldots\ldots (4a)$$

$$\Rightarrow m^2 = \frac{as + 2b^2 - 3ac}{2}$$

$$bs - mn = ad \;\; \ldots\ldots\ldots (4b)$$

$$\Rightarrow mn = bs - ad$$

$$s^2 - n^2 = ae \;\; \ldots\ldots\ldots (4c)$$

$$\Rightarrow n^2 = s^2 - ae$$

now $(mn)^2 = m^2n^2 \Rightarrow$

$$(bs - ad)^2 = \left(\frac{as + 2b^2 - 3ac}{2}\right)(s^2 - ae)$$

Simplifying and solving this equation for one value of s with trial and error method or as it will be cubic equation in s we can use cardano method to find one real value of s, using that find value of m & n.

Then using $eq(*)$ we can have two quadratic equations and hence we can solve the main bi-quadratic equation.

Exercise

1. Solve the equation $x^4 + 6x^3 + 7x^2 + 18x - 8 = 0$
2. Solve the equation $x^4 + 3x^2 + x^2 - 1 = 0$
3. Solve the equation $x^4 - 10x^2 - 20x - 16 = 0$
4. Solve the equation $x^4 + 2x^3 + 14x + 15 = 0$
5. Solve the equation $x^4 - 3x^2 - 6x - 2 = 0$

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